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Welcome to the fifth issue of Quant*News*.

Pricing Generalized Caps and Floors

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Interest rate caps and floors place an upper or lower limit on the interest payments that accrue within certain time periods in the future and are paid at the end of each period. The price of a cap or floor is the sum of the prices of the caplets or floorlets it contains, one for each time period.

Let us call δ the size of the accrual period of a caplet and τ the tenor of the index rate that determines the interest payment due at settlement date t_s. The index rate f is unknown as of now, i.e. at pricing time t_0. It is fixed at a future reset date t_r, with $t_0 < t_r < t_s$, and it is capped by the rate $\bar{R}$. The payoff occurs at t_s and the caplet price is the expectation of this payoff discounted back to now. Formally, and with r the short rate,

$$C = \delta\, E_{t_0} \left[\exp\left(- \int_{t_0}^{t_s} du\, r(u) \right) \max(f - \bar{R}, 0) \right].$$

Pricing caps is simple when the index rate naturally "matches" the caplet, i.e. $\tau = \delta$ and $t_r + \tau = t_s$. In this case the present value of the floating payment ($\delta\, f$) is *certain* and equal to its forward value. The uncertainty in the caplet price comes from the present value of the fixed payment ($\delta\, \bar{R}$). Interest rate models price such simple caplets as put options on zero coupon bonds. Conversely, simple caplets and floorlets can be used to calibrate to market the models. Caps where the starting and ending dates corresponding to the index rate do not match the affiliated caplets' accrual periods will be called "generalized."

Typical examples of generalized caps occur in adjustable rate mortages (ARMs). There, the borrower pays monthy interest based on an index rate that resets once or twice a year. The floating rate is typically either a regional cost of funds index or a treasury yield with tenor of 2 to 10 years, usually quoted with semiannual compounding. The ARM is often structured with a cap rate that limits the exposure of the borrower to large index rate increases. Pricing such embedded caps requires analysis beyond the simple case described earlier.

To price generalized caps and floors within a given interest rate model one has to compute the stochastic process followed by the floating payment. This is in addition to the process followed by the fixed payment, which is the same here as in the simple case. The larger the mismatch between the index rate and the underlying caplet the stronger the deviation of the floating payment expectation from its forward value.

Such an analysis has been carried out for the Hull-White model and has been implemented in a series of functions in QuantTools and ***@nalyst***[1]. Note that in addition to the time mismatch between the index rate and the caplet we also allow arbitrary compounding convention for the rate while keeping the caplet at simple interest. It turns out that the distribution of floating payments at reset time is lognormal but with drift and volatility depending on the index rate's tenor and compounding frequency. Once such a distribution is constructed, the caplet can be priced as a spread option between the floating and fixed payments. This is still a one-dimensional integration problem since the underlying model is one-factor.

To get a taste of the results from this analysis we consider the following example: A caplet with cap rate at 6% covers the accrual period between 1–January–2001 and 1–April–2001. The term structure as of 1–January–2000 implies constant quarterly forward rates for all maturities at 6%. The caplet is at the money and for zero volatility it is worth zero basis points for all rate tenors. We study the price of this caplet as a function of the tenor of the index rate in the Hull-White model with speed of mean reversion $\kappa = 0.01$.

It is instructive to compare the generalized caplet price C with the plausible answer

$$\frac{\delta}{\tau}\frac{P(t_0, t_s)}{P(t_0, t_r + \tau)} C_{\text{simple}} \, .$$

C_{simple} is the price of an at-the-money simple caplet that also starts on 1–Jan–2001 (t_r), but is settled τ years afterwards instead of 0.25 years (δ) afterwards. This price is scaled by the factor δ/τ. It is also scaled by the ratio of the discount factors at t_s and $t_r + \tau$ to account for the fact that the quarterly caplet pays at t_s whereas the simple caplet pays at $t_r + \tau$. In figure 1 we plot both the model price C and the scaled simple caplet of the above equation as a function of the tenor τ for various values of the Hull-White volatility parameter σ. The simple caplet is priced using the same model parameters.

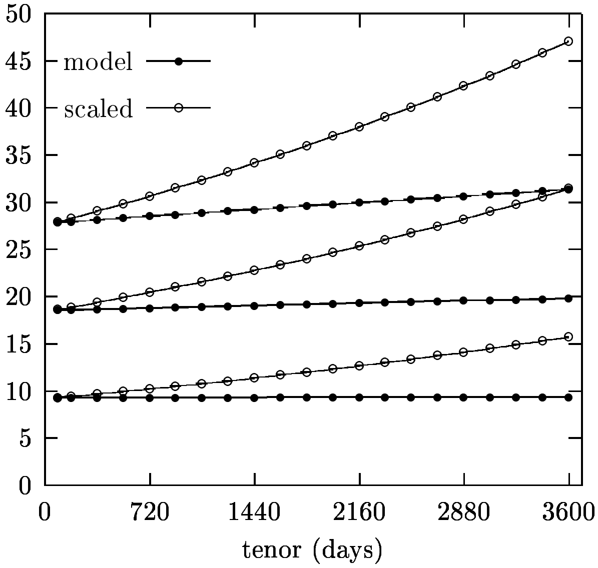

Figure 1: Caplet price (in bps) vs rate tenor. From bottom to top the curves are for $\sigma = 0.01$, 0.02 and 0.03.

The generalized caplet price increases only mildly with the tenor whereas the scaled price curves are much steeper for high volatilities. This can be understood as follows. The floating payment drifts upwards with increasing tenor and this leads to an increase in the generalized caplet's price. At the same time, though, the floating payment diffuses in a totally correlated fashion to the fixed payment. In terms

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[1] to be released later this month.

of a binomial tree setup this means that for those nodes where the fixed payment has low present value so does the floating payment. The combined effect is a milder increase in the caplet's price relative to what one would expect if the floating payment were constant.

Similar effects are observed for the floorlet, as shown in figure 2. The scaled prices are the same as in figure 1 since both the simple caplet and the simple floorlet are at the money and the diffusion of the present value of the fixed payment is symmetric around its mean value. The generalized floorlet, however, falls in value with increasing tenor because the floating payment both drifts to higher values and, at the same time, it diffuses in a totally correlated manner to the fixed payment. Drift and diffusion now work in the same direction and the fall in the generalized floorlet's price is steeper than the increase in the generalized caplet's price we saw earlier.

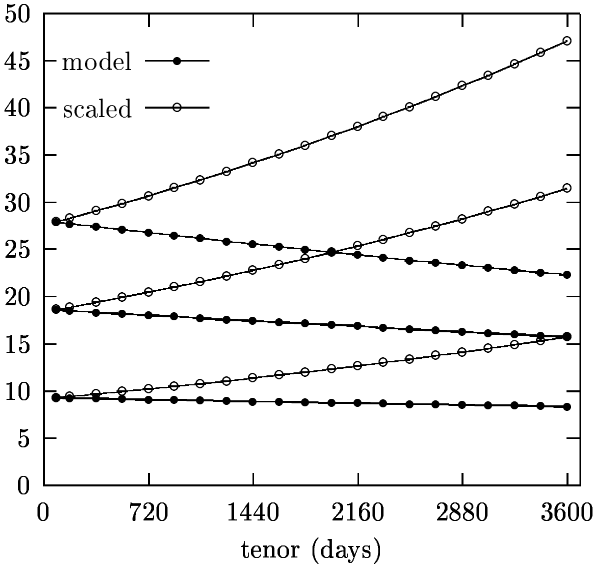

Figure 2: Floorlet price (in bps) vs rate tenor. From bottom to top the curves are for σ = 0.01, 0.02 and 0.03.

These effects become stronger as the reset time moves further into the future and the time interval for the evolution of the floating payment increases. This is quite important for adjustable rate mortgages where the cap provisions stay effective for up to thirty years from origination time.

In conclusion, the price of generalized caps and floors exhibits interesting behavior that cannot be readily reproduced from what is known in the simple case. A full analysis within the model is required. We believe that the features described here are quite generic and will also show up in other interest rate models.

Stocks Drop; Call Options Rise

When stock prices drop, the prices of call options on the stock usually drop as well. This is both intuitive and in line with option pricing theory (Δ is positive for call options). However, in some cases, a large drop in the stock price can actually cause the value of a call option to rise.

The price drop can cause the option price to rise because the new volatility of the stock may be significantly higher because of the large price movement. For certain options, the increased volatility will be a more significant effect than the direct result of the price drop: i.e. κ effects may be more important than Δ effects.

Consider a (relatively) concrete example: a stock trading at 100 with a historical six month volatility of 24%. In one day the stock drops to 80. The new six month volatility is 40%. Examine the price of six month calls struck at 140. If we assume that these options trade with values given by the historical volatility, the value of the call is \$0.26 with an underlying of 100 and 24% volatility and $0.33 with an underlying of 80 and 40% volatility. Thus a 20% drop in the price of the stock results in a 27% increase in the value of the call option.

FASB 133 in Brief

FASB 133 is one of the most far ranging and controversial accounting rule changes in recent memory. Slated for implementation for most entities in the first quarter of 2001, the new rule will require all derivative instruments to be marked to market for the first time, with unrealized gains and losses related to changes in derivative values recognized in earnings in the period of change. Previously, derivatives were frequently carried at various values that did not require reference to current market levels, such as at amortized cost, or not carried on the balance sheet at all.

Unrealized gains and losses on many other asset classes are not taken directly into earnings, and some hedgeable exposures (e.g. to currency and interest rates) do not appear on the balance sheet at all. Thus, the requirement that even unrealized derivative gains and losses be taken directly into earnings in the period in which they occur would cause unnecessary volatility in earnings from one period to the next when derivatives are used as hedges of these assets and exposures.

To help to reduce this undesirable earnings volatility, FASB 133 includes elaborate hedge accounting provisions that, in effect, allow entities to defer recognition of a portion of the gains and losses on derivatives designated as hedges, generally until offsetting gains and losses are realized on the hedged exposures. These provisions require that the entity define linkages between derivatives and hedged exposures at the time the derivative is acquired, and require the entity to define the risk being hedged. For example, if a corporate bond is hedged with a Treasury bond futures contract, then the entity might designate interest rate risk as the risk being hedged.

Only the portion of derivative gains and losses that is *effective* in hedging changes in the value of the hedged item can be deferred. The determination of the amount of the effective portion involves attributing changes in the hedge and the hedged investment to the risk being hedged. In the example above, that would mean distinguishing interest-rate-related changes in the price of the corporate bond from those related to changes in liquidity and credit, or in the case of an option, distinguishing interest-rate-related changes in the option premium from those related to volatility and time decay.

For example: suppose a corporate bond is being hedged with Treasury bond futures. Assume the market price of the bond increases by $6.00, of which $5.00 is attributed to interest rate moves and $1.00 is attributed to credit improvement (a model must already be in place to assign this distribution). If the futures position declines in value by $5.00, that loss is offset by the corresponding gain on the underlying bond. Only the $1.00 due to credit is accounted for as "other income."

If the futures position declined by $4.00 (under-hedged), this loss could be offset by the increase in the bond value, and $2.00 would be added to other income. If the futures position declined by $6.00 (over-hedged), only $5.00 could be offset. If the position declined by less than $4.00 or more than $6.50, the change is outside the 80%–125% range needed for a hedge to be considered effective, and the connection between the underlying asset and the hedge is broken.

Quantitative Finance: The Millennium in Review

Here's a selection of important events in quantitative finance over the past 1000 years.

- c. 1020 Massive printing of paper money causes inflation in China
- 1156 First known FX contract
- 1171 Bank of Venice, the first central bank, is founded
- 1397 Medici Bank founded
- 1500's checks enter widespread usage in Holland
- 1553 Muscovy Company, the first joint stock company, is founded in England
- 1600's Invention of Calculus

  Without Calculus we might be able to compute option prices but we wouldn't have the Greeks!
- 1637 Tulip bulb market crashes
- 1776 Adam Smith's *The Wealth of Nations* is published
- 1792 NYSE founded
- 1811 New York passes first general limited-liability law
- 1848 CBOT founded
- 1865 Futures contracts traded on the CBOT
- 1876 London Metal Exchange founded
- 1896 Dow Jones Industrial Average first published
- 1913 U.S. Federal Reserve System created
- 1929 Stock market crashes; great depression follows
- 1934 SEC created
- 1944 Bretton Woods Agreement creates World Bank and IMF
- 1970 First Ginnie Mae MBS issued
- 1972 DJIA hits 1000
- 1973 The Black-Scholes Model
- 1973 CBOE created
- 1983 Freddie Mac issues first CMO
- 1986 *Numerical Recipes* published
- 1986 TechHackers founded by Atul Jain and Michael How
- 1990 First Brady bonds are issued
- 1997 Merton and Scholes win Nobel Prize in Economics
- 1999 EMU creates the euro
- 1999 DJIA hits 10000

In Conclusion...

We hope you've enjoyed issue 5 of Quant*News*. Comments, submissions, and other requests should be sent to your editor at sjanowsky@thi.com.